Holonomy of Combinatorial Surfaces

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Outline

Triangulated Surfaces and Symmetry

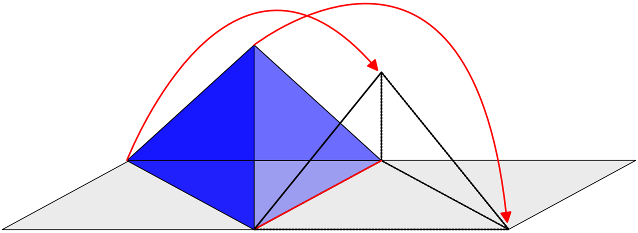
A *triangulated surface* $S = (V, F)$ consists of a set V of vertices and a set F of faces, each of which is a set of 3 vertices.

A *symmetry* of a triangulated surface $S = (V, F)$ is a face-preserving bijective map from V to itself. Symmetries of S form a group—the *automorphism group* of S, denoted $\mathrm{Aut}(S)$.

Rolling Triangulated Surfaces

Given two surfaces S and S', we can place S' on S in many different configurations.

We focused on rolling surfaces S' that are *maximally symmetric*—i.e. surfaces where every configuration of S' on a fixed face f of S corresponds to a symmetry of S'.



Loops and the Holonomy Group

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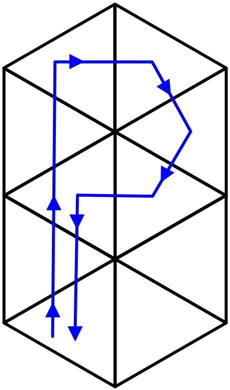
The *holonomy group* $\mathrm{Hol}_f(S', S)$ of S' over S based at face f is the image of ϕ_f.

Lassos

Lassos are a special type of loop. The holonomy induced on S' by rolling along a lasso depends only (up to conjugation) on the vertex it loops around.

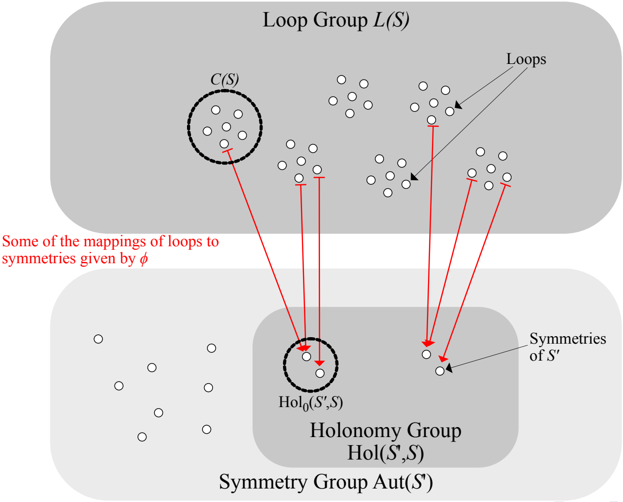


Contractible Loops & Restricted Holonomy

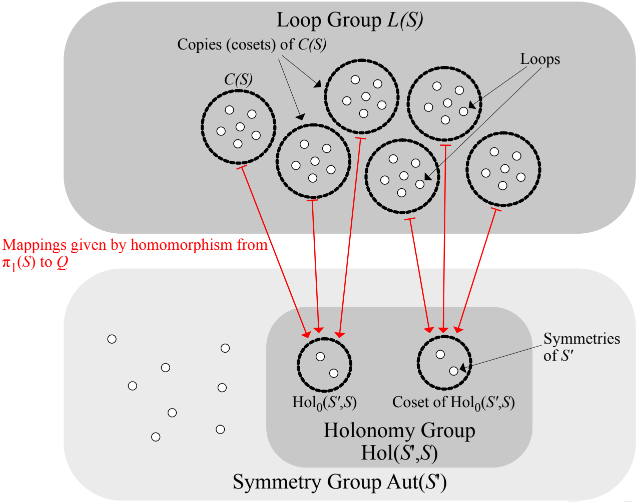
The subgroup $C(S) \trianglelefteq L(S)$ generated by lassos is the contractible loop group based at f.

The restricted holonomy group $\mathrm{Hol}_0(S', S)$ is the subgroup of the holonomy group containing the symmetries of S' that can be induced by rolling along a contractible loop ($\mathrm{Hol}_0(S', S) = \phi(C(S))$).

Structure of the Holonomy Group



Structure of the Holonomy Group

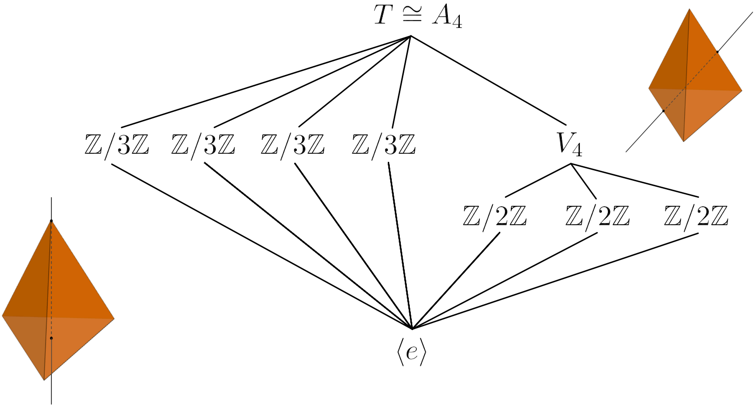


Computing Holonomy Groups

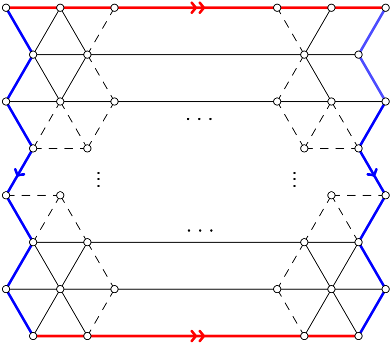
This gives a method to compute the holonomy group:

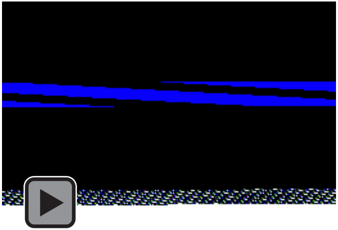
1. Find the restricted holonomy group.
2. For each coset of $C(S)$ in $L(S)$, find the coset of the contractible holonomy group whose elements (symmetries) those loops induce on S'.
3. Repeat until we have enough information to determine the holonomy group; usually, examining just 2 or 3 cosets of $C(S)$ is enough.

The Tetrahedral Group



Assembly of Torus





Holonomies of Simple Setups

Setup (R,T,L)	Holonomy	Vertex Permutations
$(0,0,0)$	Trivial	-
$(1,0,0)$ Left	$\mathbb{Z}/2\mathbb{Z}$	(1,3)(2,4)
$(2,0,0)$	$\mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)
$(3,0,0)$ Right	$\mathbb{Z}/2\mathbb{Z}$	(1,3)(2,4)
$(0,2,0)$	$\mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)
$(0,0,2)$	$\mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)
$(0,1,3)$	$\mathbb{Z}/2\mathbb{Z}$	(1,2)(3,4)

Table: Holonomies of Basic Setups of Torus

Results for Tori

Setup (R,T,L)	Holonomy	Combinations	Permutations
$(1,1,1)$ Left	$\mathbb{Z}/2\mathbb{Z}$	$((1,3)(2,4))^2$	(1,3)(2,4)
$(1,1,3)$ Right	$\mathbb{Z}/2\mathbb{Z}$	$((1,2)(3,4))^2$	(1,2)(3,4)
$(0,2,2)$	Trivial	$((1,4)(2,3))^2$	-
$(2,2,0)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3)^2$	(1,4)(2,3)
$(2,0,2)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3))^2$	(1,4)(2,3)
$(2,3,3)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)(1,3)(2,4)	$\mathbb{V}$

Table: Holonomies of Combination Setups of Torus

Tori with Special Vertices

Currently we have: Trivial, $\mathbb{Z}/2\mathbb{Z}$ & $\mathbb{V}$. We still need: $\mathbb{Z}/3\mathbb{Z}$ & $\mathbb{A}_4$.

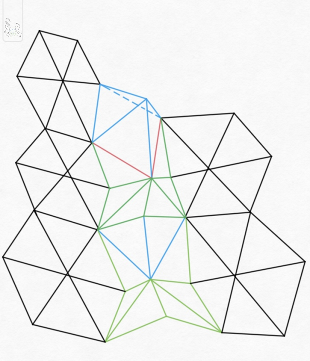
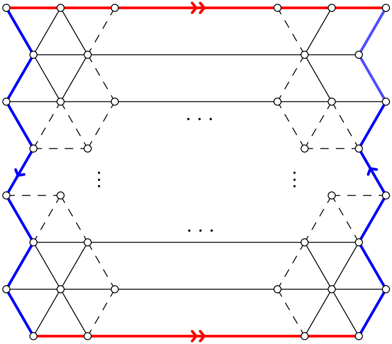
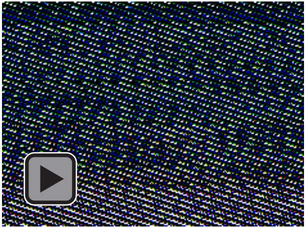


Figure: Bordering Order 6 Vertices with Order 4 and 8

Assembly of Klein Bottle





Results for Klein Bottle

Setup (R,T,L)	Holonomy	Permutations
$(0,0,0)$ Left	$\mathbb{Z}/2\mathbb{Z}$	(2,3)
$(0,0,0)$ Right	$\mathbb{Z}/2\mathbb{Z}$	(1,4)
$(1,0,0)$	$\mathbb{Z}/4\mathbb{Z}$	(1,3,4,2)
$(3,0,0)$ Right	$\mathbb{Z}/4\mathbb{Z}$	(1,2,4,3)
$(0,0,2)$	$\mathbb{Z}/4\mathbb{Z}$	(2,3)
(0,1,3)	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{V}$
$(2,3,3)$	$\mathbb{D}_4$	$\mathbb{D}_4$

Table: Holonomies of Basic Setups of Klein Bottle

Restricting the holonomy to subgroup of $\mathbb{Z}/3\mathbb{Z}$

Assumption: S is connected, closed, and orientable.

In this section, we explore a different method to identify two specific holonomy groups: the trivial and $\mathbb{Z}/3\mathbb{Z}$, when we roll the tetrahedron over the surface S.

Intuition

What does it mean to have trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy?

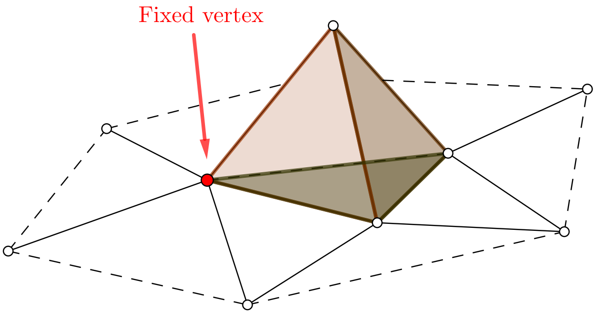


Figure: Having a trivial or $\mathbb{Z}/\mathbb{Z}_3$ holonomy is the same as having a fixed vertex

Intuition

So we want to find whether there is a structure of our surface that can indicate whether a vertex of the surface can "fix" the tetrahedron's vertex?

To do that, given a vertex x of the tetrahedron, we want to look at which vertices of the surface can x touch.

Observation

Given a face of the surface, we can see that there are 3 other faces that are adjacent to this face, and if x stands on any of the red vertices, it's possible for x to reach the other two.

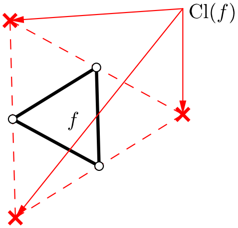
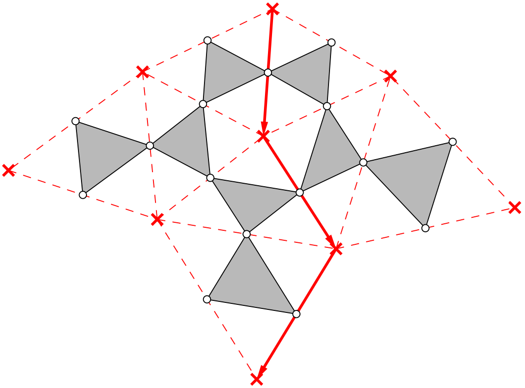


Figure: A vertex x touching any red vertex can reach the other two by sufficient rollings

By this, we can define an equivalence relation saying that two vertices are equivalence if and only if we can go from one to other through a finite sequence of neighborhoods.



Isolated vertex

Definition

A vertex is *isolated* if it's not equivalent to any of its neighbors.

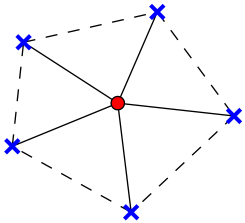


Figure: Isolated vertex is not equivalent to all of its neighbors

Theorem of restricting the holonomy group

Theorem

For S orientable, the holonomy group is a subgroup of $\mathbb{Z}/3\mathbb{Z}$ if and only if S has an isolated vertex.

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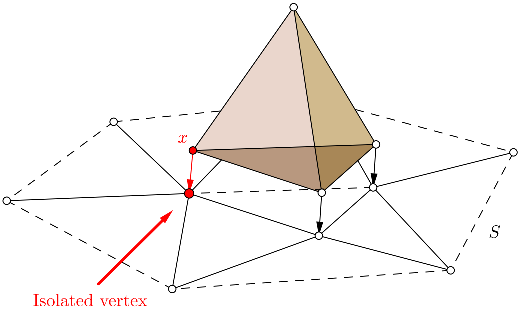


Trivial and $\mathbb{Z}/3\mathbb{Z}$ criteria

Theorem

For S orientable, S has trivial holonomy iff it has all vertices isolated, and it has $\mathbb{Z}/3\mathbb{Z}$ holonomy iff it has both isolated and non-isolated vertices.

Application: subdivision

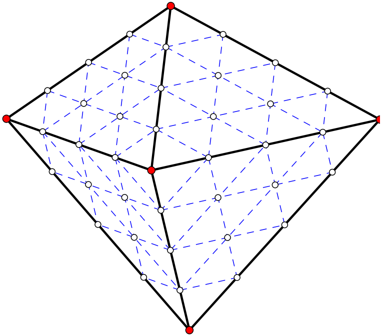


Figure: 4−subdivision

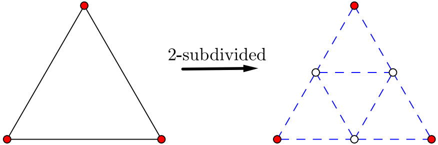
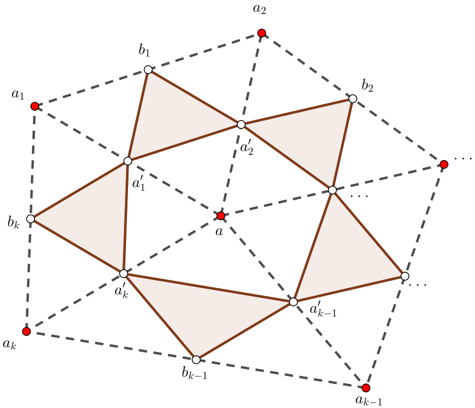


Figure: 2−subdivision

Even subdivision holonomy

Theorem

2−subdivision has holonomy trivial or $\mathbb{Z}/3\mathbb{Z}$.



$\mathbb{Z}/3\mathbb{Z}$ holonomy construction using subdivision

Theorem

If the original surface has a vertex with degree not divisible by 3, then 2-subdivision of it has holonomy $\mathbb{Z}/3\mathbb{Z}$.

Conclusion

- We developed tools to find holonomy group by first using the concept of contractible holonomy group and combinatorial fundamental group.
- We then applied this method to compute holonomy group of tori, specifically for the tetrahedron case, and gave examples of surfaces with full symmetry group, Klein-four group, and $\mathbb{Z}/2\mathbb{Z}$ holonomies.
- We then give a method to check whether we have the trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy, and then construct an example of $\mathbb{Z}/3\mathbb{Z}$ holonomy using subdivision, verified by this method.

Future work

- Apply or generalize our methods to more complicated surfaces
- Examine other complex problems.
- Develop efficient programs to simulate and solve these problems.

Thank you for listening!