Holonomy of Combinatorial Surfaces

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1 Introduction

The concept of holonomy explores how smooth surfaces interact against each other by "rolling" one surface over another through arbitrarily closed loops around a base point. In our project, we focused on the discrete setup of this problem, in particular, we referred to triangular surfaces where we put an interaction between two surfaces by choosing two different faces of each one and matching vertices between them. This choice of two touching faces and the matching between their vertices define a position between the two surfaces. So our general goal of understanding the interaction between two surfaces is to figure out how the position changes after rolling the surfaces over each other, what are all possible positions we can get after rolling in arbitrary loops.

To understand the big picture, we started by making observations for a simple, highly symmetric structure - the tetrahedron - and examining what possible holonomy groups we can get from rolling it over different surfaces. Along with that, we built up our understanding of loop structure on a given surface, conjecturing about what features that loops may have and how those features give us computational tools to break down the holonomy group. Then, we introduced the concepts of contractibility and combinatorial fundamental group, brought from a perspective of algebraic topology. It helped us slightly reduce the complexity of calculating the holonomy groups. We then managed to develop mathematical tools using those ideas and apply them to compute the holonomy of triangular tori by looking at the generating elements induced by non-contractible loops. This gave us a classification of tori's holonomies that vary from trivial, $\mathbb{Z}/2\mathbb{Z}$, V_4, to full tetrahedral groups. In the end, to identify surfaces that have the $\mathbb{Z}/3\mathbb{Z}$ holonomy, we introduced an entirely different method by dealing with the structure on the vertices of the surface. This method can identify exactly trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy for the orientable surface. Then, we used subdivided surfaces to apply this method and showed an example of a surface with $\mathbb{Z}/3\mathbb{Z}$ holonomy.

While the smooth holonomy problem has already been studied thoroughly, combinatorial holonomy is still a new approach that very few people has studied. By looking at this problem discretely, it opens many interesting problems to think about, as well as many remarkable results. By understanding the combinatorial holonomy, we can draw a connection to the smooth holonomy problem.

That connection is helpful for us to develop a computational tool for the holonomy of arbitrary smooth surfaces, by approximating it with a discrete version that makes it plausible to implement and simulate using algorithms.

2 Background

This section provides a general synopsis of main terms and ideas used throughout the research paper.

2.1 The Tetrahedral Group

We spend much of this paper working with the symmetries of the tetrahedron, so we begin with a brief overview of the structure of the tetrahedron's isometry group.

There are 12 rotations of the tetrahedron: for each of the four faces of the tetrahedron, there are three rotations holding that face fixed. There are an additional 12 symmetries that combine a reflection and a rotation. The 8 pure rotations about an axis passing through a vertex and the center of the opposite face form one conjugacy class and all have order 3. The 3 pure rotations about an axis passing through the midpoints of two non-adjacent edges form another conjugacy group and all have order 2. The 6 pure reflections of the tetrahedron, which each exchange two vertices and leave two vertices fixed, form an additional conjugacy class whose elements have order 2. The remaining 6 non-identity elements, all of which have order 4 and are the product of a pure rotation and a reflection involving the vertex fixed by the rotation, form a conjugacy class containing precisely the elements of order 4. Finally, the identity lies in its own conjugacy class.

Thus, the tetrahedron's isometry group can be seen as a group of 24 permutations of the four vertices of the tetrahedron. But there are only 24 possible such permutations, so the isometry group is actually isomorphic to S_4 the symmetric group on four letters.

Often, we will focus on the subgroup of the tetrahedron's isometry group containing only the pure rotations, which are the "orientation-preserving" isometries of the tetrahedron. As a subgroup of order 12, this subgroup must be isomorphic to A_4, the group of even permutations on four letters, which is the only order-12 subgroup of S_4.

The subgroup structure of A_4 is particularly relevant. A_4 contains three subgroups of order 2, each of which is generated by one of the rotations of order 2. A_4 also contains a subgroup of order 4 containing all such rotations; this subgroup contains no element of order 4, so it is isomorphic to V_4, the Klein four-group. Notably, A_4 contains no proper subgroup of order greater than 4, so any subgroup containing more than 3 nontrivial group elements must be the whole group, and any subgroup containing both an element of order 2 and an element of order 3 must be the whole group.

2.2 Triangulated Surfaces

Definition 2.1 (Triangulated surface)**.** A *triangulated surface* $S = (V, F)$ is a pair of nonempty sets: V, the set of *vertices* of S, and a set F, the set of *faces* (or *triangles*) of S. These are required to satisfy three axioms:

1. Each face $f \in F$ is of the form $f = \{v_1, v_2, v_3\}$, where $v_i \in V$ are distinct elements.
2. The intersection of two distinct faces $f_1 \neq f_2$ in F contains at most two elements. (When $f_1 \cap f_2 = \{v_1, v_2\}$ with $v_1 \neq v_2$, we say that f_1 and f_2 are *adjacent*.)
3. For every $v \in V$, the set of faces $f \in F$ that contain v is non-empty and it is possible to arrange these faces in a sequence $(f_1, f_2, ..., f_\delta)$ such that f_i and f_{i+1} are adjacent for $1 \leq i < \delta$. (The number δ is called the *degree* of v, and is also often denoted $\delta(v)$. If, in addition f_δ is adjacent to f_1, we say that v is an *interior vertex*, otherwise, we say that v is a boundary vertex.)

We also define the set E of *edges* of S to be the set of 2-element subsets $e = \{v_1, v_2\}$ that are contained in some face $f \in F$. In particular, each face $f = \{v_1, v_2, v_3\}$ contains three edges: $\{v_2, v_3\}$, $\{v_3, v_1\}$, and $\{v_1, v_2\}$. An edge of S that belongs to only one face of S is said to be a *boundary edge*, otherwise, if it belongs to two distinct faces of S, it is said to be an *interior edge.*

If all the edges of a surface are interior, then so are all the vertices, and we say S has 'no boundary'.

Definition 2.2 (Connectedness)**.** A triangulated surface $S = (V, F)$ is said to be connected if, for any two vertices $v, v' \in V$, there exists a sequence of vertices $v = v_1, v_2, \cdots, v_k = v' \in S$ such that $\{v_i, v_{i+1}\}$ is an edge for all $1 \leq i < k$.

We can also refer connectedness to faces instead of vertices. Namely, a surface $S = (V, F)$ is *face-connected* if for any $f, f' \in F$, there exists a sequence of S-faces $f = f_1, f_2, \cdots, f_k = f$ such that f_i is adjacent to f_{i+1} for $1 \leq i < k$. It can be shown that these two connectedness notions are equivalent: a surface is connected if and only if it is face-connected.

Definition 2.3 (Automorphism, automorphism group)**.** For a triangulated surface $S = (V, F)$, a bijection $\alpha : V \to V$ is an automorphism of S if $\{v_1, v_2, v_3\} \in F$ iff $\{\alpha(v_1), \alpha(v_2), \alpha(v_3)\} \in F$. The set of automorphisms of S forms a group under composition, which we denote $\mathrm{Aut}(S)$.

For any connected surface S, if some automorphism $\alpha \in \mathrm{Aut}(S)$ fixes all the vertices of a single face of S, then α must be the identity automorphism. This is because, once α fixes all the vertices of one face, it must also fix all vertices of the adjacent faces, so the connectedness of S guarantees that it in fact fixes all vertices of all faces of S. Thus, for a connected triangulated surface, seeing where an autoorphism maps a single face is sufficient to completely determine the automorphism. This means $|\mathrm{Aut}(S)| \leq 6|F|$, since a fixed face of S can, at most, be mapped to each face of S in six ways.

Definition 2.4 (Maximally symmetric)**.** A triangulated surface $S = (V, F)$ with $|\mathrm{Aut}(S)| = 6|F|$ is *maximally symmetric.*

Definition 2.5 (Orientation)**.** Given a face $f = \{v_1, v_2, v_3\} \in F$ of a triangulated surface S, an *orientation* of f is an ordering of the vertices of f (i.e. clockwise or counterclockwise). This ordering depends only on the relative positions of the vertices in the ordering, not on the specific order in which they are listed; that is, the orderings (v_1, v_2, v_3), (v_2, v_3, v_1), and (v_3, v_1, v_2) are all equivalent and can be denoted $[v_1, v_2, v_3]$. Similarly, an orientation of an edge $e = \{v_1, v_2\}$ of S is an ordering of the two vertices of e. For each edge, there are two possible orientations: $[v_1, v_2]$ and $[v_2, v_1]$.

An *orientation of* $S = (V, F)$ is a choice of orientation for each face $f \in F$ such that, whenever two faces share an edge (i.e. whenever $f \cap f' = \{v_1, v_2\} = e$ for some $f, f' \in F$), the two faces f and f' induce opposite orientations on e. If it is possible to define an orientation on S, then S is orientable.

Definition 2.6 (Position)**.** Given two triangulated surfaces $S = (V, F)$ and $S' = (V', F')$, a triple of matchings

$$p = \{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\}$$

where $f = \{v_1, v_2, v_3\} \in F, f' = \{v_1', v_2', v_3'\} \in F'$, is called a *position.* The set of all possible position given a pair of surfaces S, S' is $P(S, S')$.

Note that for a position, it's not only the two faces of S, S' that are important, but it is also the exact matching of the vertices between them.

Definition 2.7 (Connected by a single roll)**.** Consider the position

$$p = \{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\}$$

and assume that $\{v_1, v_2\}, \{v_1', v_2'\}$ are internal edges of S, S', respectively, i.e., there exist faces $f = \{v_1, v_2, v_4\} \in F, \{v_1', v_2', v_4'\} \in F'$. Then, the position

$$\hat{p} = \{(v_1, v_1'), (v_2, v_2'), (v_4, v_4')\}$$

is said to be *connected to p by a single roll.*

Note that for two surfaces with no boundaries S, S', a position is connected to exactly 3 other position by a single roll.

2.3 Homotopy and Fundamental Groups

Our investigation centers on the combinatorial notion of a triangulated surface, but we begin by stating the traditional definitions of homotopy and the fundamental group for a general topological space (taken from Hatcher) before proceeding to state our combinatorial analogues. A proof of the compatibility of the traditional and combinatorial formulations of the fundamental group is given in the appendix.

Definition 2.8 (Homotopy)**.** A *path* in a topological space X is a continuous map $f : [0,1] \to X$. A *homotopy* of paths in X is a family of maps $f_t : [0,1] \to X$ for $t \in [0,1]$ such that (a) $f_t(0)$ and $f_t(1)$ are independent of t, and (b) the map $F : [0,1]^2 \to X$ given by $F(s,t) = f_t(s)$ is continuous. If a homotopy exists between two paths f and g, then those paths are *homotopic* and we write $f \simeq g$. Moreover, $\simeq$ is an equivalence relation.

Two paths f and g can be composed to obtain a product path gf that follows f and then g so long as $f(1) = g(0)$. If f and g are loops with the same basepoint, i.e. $f(0) = f(1) = g(0) = g(1) = x_0$, then the compositions fg and gf are guaranteed to exist.

Definition 2.9 (Fundamental group)**.** The *fundamental group* $\pi_1(X, x_0)$ of a space X is the group of equivalence classes under homotopy of loops based at x_0 under the operation of path composition. The group operation is well-defined on these equivalence classes. Furthermore, the fundamental group, up to isomorphism, is independent of the basepoint for path-connected spaces. Hence, if X is path-connected, we can unambiguously write $\pi_1(X)$ for the fundamental group of X.

In the following section we define separately the notion of a fundamental group of a triangulated surface, but we show that our combinatorial definition is agrees with the traditional definition given above.

Given a triangulated surface $S = (V, F)$ without boundary, we can define a combinatorial analogue of the conventional fundamental group by considering loops along the faces of S. Once we also define the notion of a combinatorial holonomy group, this combinatorial fundamental group will help us prove an important result on the structure of the combinatorial holonomy group.

Definition 2.10 (Path, loop, loop group)**.** A path along S is a sequence of faces $f_i \in F$, $1 \leq i \leq n$, such each f_i is adjacent to both f_{i-1} and f_{i+1}. A loop is a path where $f_1 = f_n$. In other words, a loop on S is a sequence of faces, starting and ending at the same face, that another triangulated surface could be rolled along. Given a fixed "base face" $f \in F$, the set of loops in S beginning and ending at f forms a group under concatenation of loops—the loop group—which we denote by $L_f(S)$. The operation of concatenation of loops is denoted by $\star$.

To be entirely precise, $L_f(S)$ is actually a group of equivalence classes of loops, where two loops are considered equivalent if one can be obtained from the other by adding or removing backtracking.

Definition 2.11 (Backtracking-equivalence)**.** If $l = (f_1, f_2, ..., f_n)$ and $l' = (f'_1, f'_2, ..., f'_m)$ are loops (i.e. $f_1 = f_n$ and $f'_1 = f'_m$), then we consider l and l' equivalent if one of the loops (say, l) can be obtained from the other (in this example, l') by replacing some portion of l of the form $(f_i, f_{i+1}, f_{i+2} = f_i)$ with (f_i). For example, the loops

$$l = (f_1, f_2, f_3, f_4, f_3, f_5, f_1)$$

and

$$l' = (f_1, f_2, f_3, f_5, f_1)$$

are equivalent (assuming the listed faces are adjacent as needed). It is straightforward to confirm that the group operation of concatenation is well-defined on these equivalence classes of loops.

Theorem 2.1. *Assuming our surface S is path-connected (i.e. that there exists a path between any pair of faces of S), the group $L_f(S)$ is independent of the base face up to isomorphism.*

Proof. If $f, g \in F$ are faces and p is a path from f to g, then there is a group homomorphism $L_f(S) \to L_g(S)$ given by $l \mapsto p \star l \star p^{-1}$. In particular, this homomorphism is surjective, because any loop $l' \in L_g(S)$ is mapped to by the loop $p^{-1} \star l' \star p \in L_f(S)$. This also shows that the homomorphism has an inverse homomorphism $L_g(S) \to L_f(S)$ given by $l \mapsto p^{-1} \star l \star p$, so it is an isomorphism—i.e. $L_f(S) \cong L_g(S)$ for any $f, g \in F$. $\square$

We can now unambiguously write $L(S)$ for the group of loops on S up to isomorphism, so we will omit the base face in our notation except when it may be relevant to the argument at play.

A lasso is a special and important type of loop in $L(S)$: specifically, a lasso is a loop that follows some path, loops directly around a single vertex, and then returns along the original path. This is formalized in the following definition.

Definition 2.12 (Lasso)**.** If p is a path from f_1 to f_n, $v \in f_n$ is an interior vertex of f_n, and l is the loop around v beginning and ending at f_n whose existence is guaranteed by property (3) of the definition of a triangulated surface (or that loop's inverse), then the loop $p^{-1} \star l \star p$ is a lasso based at f_1.

Definition 2.13 (Contractible loop group, contractibility)**.** The subgroup $C_f(S)$ of $L_f(S)$ generated by the set of all lassos (based at f) in $L_f(S)$ is called the group of contractible loops on S based at f. Like $L(S)$, $C(S)$ is unique up to isomorphism by Theorem 2.1, so the base face will often be omitted for convenience. The loops in $C(S)$ are the contractible loops based at f.

This definition of contractibility agrees with the standard definition of null-homotopic loops in a continuous sense; a combinatorial loop on S is contractible iff the same loop, considered instead as a continuous loop in the underlying topological space of S, is null-homotopic. A proof of this result is given in the appendix. Similarly, we can consider two loops homotopic if their continuous analogues in the underlying topological space are homotopic.

Theorem 2.2. *$C(S)$ is a normal subgroup of $L(S)$.*

Proof. Suppose $c \in C(S)$ and $l \in L(S)$. Then lcl^{-1} is homotopic to $l\mathbf{1}l^{-1}$, which is clearly equal to the identity $\mathbf{1} \in C(S)$. Thus lcl^{-1} is null-homotopic, so $lcl^{-1} \in C(S)$. $\square$

Definition 2.14. The fundamental group of a connected triangulated surface S is defined as $\pi_1(S) := L(S)/C(S)$.

This definition agrees with the traditional notion of a fundamental group in that the fundamental group of a surface is isomorphic to the fundamental group of the surface's underlying topological space. (See the appendix for details.)

3 The Combinatorial Holonomy Group

Now, we proceed to define the combinatorial holonomy group. Let $S = (V, F)$ be a maximally symmetric triangulated surface and let $S' = (V', F')$ be a connected triangulated surface without boundary and with at least two faces. Recall that $P(S, S')$ is the set of all positions of S on S'.

There exist two natural *face projections* $\phi : P(S, S') \to F$ and $\phi' : P(S, S') \to F'$ that map a position to the face of S or S' (respectively) that position aligns with the other surface; for instance, if $p = \{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\} \in P(S, S')$, then $\phi(p) = \{v_1, v_2, v_3\} \in F$ and $\phi'(p) = \{v_1', v_2', v_3'\} \in F'$.

Now, $\mathrm{Aut}(S)$ acts naturally on $P(S, S')$ by applying to S in-place: for any $\alpha \in \mathrm{Aut}(S)$, define

$$\alpha(\{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\}) = \{(\alpha(v_1), v_1'), (\alpha(v_2), v_2'), (\alpha(v_3), v_3')\}$$

Since α permutes the vertices of S without changing which face of S' it lies on, $\alpha(\phi'(p)) = \phi'(\alpha(p))$ for all $p \in P(S, S')$. Furthermore, because S is maximally symmetric by assumption, it's clear that if $\phi'(p_1) = \phi'(p_2)$ for some $p_1, p_2 \in P(S, S')$, then there exists some $\beta \in \mathrm{Aut}(S)$ such that $p_2 = \beta(p_1)$. This means that the action of $\mathrm{Aut}(S)$ on each of the "fibers"

$$P(S, S')_{f'} = (\phi')^{-1}(f') = \{p \in P(S, S') \mid \phi'(p) = f'\}$$

is simply transitive. (Transitivity follows immediately from the fact that S is maximally symmetric; that the action is free is a consequence of the observation that an automorphism of S is completely determined by where it sends a single face.)

Before proceeding, we take a moment to state an important lemma.

Lemma 3.1. *Let $\{v_1, v_2, v_3\}, \{v_5, v_6, v_7\} \in F$ be faces of S (where $S = (V, F)$ is, as above, a connected maximally symmetric triangulated surface without boundary). Also let $v_4, v_8 \in V$ be the unique vertices such that $\{v_1, v_2, v_4\}$ and $\{v_5, v_6, v_8\}$ are also faces of S. Finally, let $\alpha_1, \alpha_2, \beta \in \mathrm{Aut}(S)$ be the unique automorphisms of S such that*

$$(\alpha_1(v_1), \alpha_1(v_2), \alpha_1(v_3)) = (v_1, v_2, v_4) \text{ and } (\alpha_2(v_5), \alpha_2(v_6), \alpha_2(v_7)) = (v_5, v_6, v_8).$$

meanwhile $(\beta(v_1), \beta(v_2), \beta(v_3)) = (v_5, v_6, v_7)$. Then $\beta \circ \alpha_1 = \alpha_2 \circ \beta$.

The proof of the lemma is a consequence of the definitions of the automorphisms and the fact that an automorphism is entirely defined by where it sends the vertices of a single face.

Now, pick two fixed faces $f'_1, f'_2 \in F'$ of S', say, $f_1 = \{v'_1, v'_2, v'_3\}$ and $f'_2 = \{v'_1, v'_2, v'_4\}$, and consider rolling S from f'_1 to f'_2 over the edge $e' = \{v'_1, v'_2\}$. This roll can be thought of as the unique mapping $\rho_{e'} : P(S, S')_{f'_1} \to P(S, S')_{f'_2}$ between fibers such that

$$\rho_{e'}(\{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3)\}) = \{(v_1, v'_1), (v_2, v'_2), (v_4, v'_4)\}.$$

The vertex v_4 is appropriately chosen as the vertex of S such that $\{v_1, v_2, v_4\} \in F$; for any position in the fiber $P(S, S')_{f'_1}$, this choice is unique, so $\rho_{e'}$ is indeed uniquely defined. Then, applying Lemma 3.1, we have the crucial relation

$$\rho_{e'}(\beta(p)) = \beta(\rho_{e'}(p)) \tag{1}$$

for all $\beta \in \mathrm{Aut}(S)$ and all $p \in P(S, S')$ (where e' is chosen appropriately).

Theorem 3.2. *Let $S = (V, F)$ be a connected, maximally symmetric triangulated surface without boundary, let $S' = (V', F')$ be a connected triangulated surface, and let $f' \in F'$ be a face of S'. Then each choice of $p \in P(S, S')_{f'}$ gives a unique holonomy homomorphism $h_p : L_{f'}(S') \to \mathrm{Aut}(S)$. Moreover, $h_{\beta(p)} = \beta \circ h_p \circ \beta^{-1}$ for all $\beta \in \mathrm{Aut}(S)$.*

Proof. Let $\lambda = (f'_1, f'_2, .., f'_d) \in L_{f'}(S')$ be a loop, where $f' = f'_1 = f'_d$. For $1 \le i < d$, let e'_i be the edge shared by both f'_i and f'_{i+1}. Now, beginning with $p = p_1 \in P(S, S')_{f'}$, we inductively define

$$p_{i+1} = \rho_{e_i}(p_i)$$

for $1 \le i < d$, so $p_i \in P(S, S')_{f'_i}$ for all $1 \le i \le d$.

Note $p_d = (\rho_{d-1} \circ \rho_{d-2} \circ \cdots \circ \rho_1)(p)$, so p_d is entirely determined by the initial position p and the choice of the loop λ. Hence, there exists a unique $\alpha \in \mathrm{Aut}(S)$ such that $p_d = \alpha(p)$, so we can define $h_p(\lambda) := \alpha$.

To show that h_p is a homomorphism, suppose $\tilde{\lambda} = (\tilde{f}'_1, \tilde{f}'_2, ..., \tilde{f}'_{\tilde{d}}) \in L_{f'}(S')$ is another loop, and consider the composition in $L_{f'}(S')$

$$\lambda \star \tilde{\lambda} = (f'_1, ..., f'_d = \tilde{f}'_1, ..., \tilde{f}'_{\tilde{d}}).$$

As before, we can inductively define a sequence of positions

$$(p_1, ..., p_d = p_d, ..., p_{d+\tilde{d}-1})$$

such that p_i and p_{i+1} are connected by a single roll for all $1 \le i < d + \tilde{d} - 1$. In particular, $p_1, ..., p_d$ are as above, and, writing $\tilde{e}_i$ for the edge shared by faces $\tilde{f}'_{i-1}$ and $\tilde{f}'_i$, we have $p_{i+1} = \rho_{\tilde{e}_{i-d+1}}(p_i)$ for $d \le i \le d + \tilde{d} - 2$. Thus,

$$\begin{aligned}
h_p(\lambda \star \tilde{\lambda})(p) = p_{d+\tilde{d}-1} &= \left(\rho_{\tilde{e}_{\tilde{d}-1}} \circ \rho_{\tilde{e}_{\tilde{d}-2}} \circ \cdots \circ \rho_{\tilde{e}_1}\right)(p_d) \\
&= \rho_{\tilde{e}_{\tilde{d}-1}} \circ \rho_{\tilde{e}_{\tilde{d}-2}} \circ \cdots \circ \rho_{\tilde{e}_1}(h_p(\lambda)(p)) \\
&= (h_p(\lambda)) \left(\rho_{\tilde{e}_{\tilde{d}-1}} \circ \rho_{\tilde{e}_{\tilde{d}-2}} \circ \cdots \circ \rho_{\tilde{e}_1}(p)\right) \\
&= h_p(\lambda) \left(h_p(\tilde{\lambda})(p)\right) = \left(h_p(\lambda) \circ h_p(\tilde{\lambda})\right)(p)
\end{aligned}$$

where the third line follows from (1).

To prove the final relation, let $\beta \in \mathrm{Aut}(S)$ and let $\tilde{p} = \beta(p)$. Then, by (1), we have that

$$\begin{aligned}\tilde{p}_2 = \rho_{e_1}(\tilde{p}) &= \rho_{e_1}(\beta(p)) \\ &= \beta(\rho_{e_1}(p)) \\ &= \beta(p_2),\end{aligned}$$

so repeated application of (1) gives $\tilde{p}_i = \beta(p_i)$, and in particular, $\tilde{p}_d = \beta(p_d)$. Thus,

$$\begin{aligned}(h_{\beta(p)}(\lambda) \circ \beta)(p) &= h_{\beta(p)}(\lambda)(\beta(p)) \\ &= \tilde{p}_d \\ &= \beta(p_d) \\ &= (\beta \circ h_p(\lambda))(p).\end{aligned}$$

Since the action of $\mathrm{Aut}(S)$ on the fiber $P(S, S')_{f'}$ is simply transitive, this implies that $h_{\beta(p)}(\lambda) \circ \beta = \beta \circ h_p(\lambda)$. Because λ was chosen arbitrarily, we conclude that $h_{\beta(p)} = \beta \circ h_p \circ \beta^{-1}$. ☐

Corollary 3.1. *The image $h_p(L_{f'}(S')) \subset \mathrm{Aut}(S)$ is a subgroup of $\mathrm{Aut}(S)$ whose conjugacy class is independent of p and f'.*

Proof. The independence of the conjugacy class from p follows immediately from the relation $h_{\beta(p)} = \beta \circ h_p \circ \beta^{-1}$. To prove independence from f', suppose $f'_1 \in F'$ is another face of S', and, as we are assuming S' is connected, let $\gamma = (f'_1, f'_2, ..., f'_n = f')$ be a path from f'_1 to f'. By Theorem 2.1, for each loop $l \in L_{f'_1}(S')$, there exists a unique loop

$$\lambda = (f' = \tilde{f}'_1, \tilde{f}'_2, ..., \tilde{f}'_d = f') \in L_{f'}(S')$$

such that $l = \gamma \star \lambda \star \gamma^{-1}$. Let e'_i be the edge between the f'_i and f'_{i+1} for $1 \leq i < n$, and let $\tilde{e}'_i$ be the edge between $\tilde{f}'_i$ and $\tilde{f}'_{i+1}$ for $1 \leq i < d$. Then, fixing some $p \in P(S, S')_{f'_1}$ we have

$$\begin{aligned}h_p(l)(p) &= h_p(\gamma \star \lambda \star \gamma^{-1})(p) \\ &= (\underbrace{\rho^{-1}_{e'_1} \circ \cdots \circ \rho^{-1}_{e'_{n-1}}}_{\gamma^{-1}} \circ \underbrace{\rho_{\tilde{e}'_{d-1}} \circ \cdots \circ \rho_{\tilde{e}'_1}}_{\lambda} \circ \underbrace{\rho_{e'_{n-1}} \circ \cdots \circ \rho_{e'_1}}_{\gamma})(p)\end{aligned} \tag{2}$$

where the second equality follows from the definition of the holonomy homomorphism h_p.

Now let $\tilde{p} \in P(S, S')_{f'}$ be the unique position such that

$$\tilde{p} = (\rho_{e'_{n-1}} \circ \cdots \circ \rho_{e'_1})(p).$$

Then we can rewrite the above equation as follows:

$$\begin{aligned}
h_p(l)(p) &= (2) \\
&= (\rho_{e_1'}^{-1} \circ \cdots \circ \rho_{e_{n-1}'}^{-1} \circ h_{\tilde{p}}(\lambda) \circ \rho_{e_{n-1}'} \circ \cdots \circ \rho_{e_1'})(p) \\
&= (h_{\tilde{p}}(\lambda))(\rho_{e_1'}^{-1} \circ \cdots \circ \rho_{e_{n-1}'}^{-1} \circ \rho_{e_{n-1}'} \circ \cdots \circ \rho_{e_1'})(p) \qquad \text{by (1)} \\
&= h_{\tilde{p}}(\lambda)(p).
\end{aligned}$$

Note that the maps ρ are bijections between fibers of the position space $P(S, S')$, whereas $h_{\tilde{p}}(\lambda)$ is an automorphism of S that acts on each of those fibers, so the use of the function composition notation is not entirely precise. Regardless, since the action of $\mathrm{Aut}(S)$ on $P(S, S')_{f_1'}$ is simply transitive, we can conclude that $h_p(l) = h_{\tilde{p}}(\lambda)$. And because there is a bijection between the loops $l \in L_{f_1'}(S')$ and the loops $\lambda \in L_{f'}(S')$, we have $h_p(L_{f_1'}(S')) = h_{\tilde{p}}(L_{f'}(S')) \subset \mathrm{Aut}(S)$, as desired. □

Definition 3.1 (Holonomy group)**.** Let S be a maximally symmetric triangulated surface without boundary and let S' be a triangulated surface without boundary. Let f' be a face of S' and let $p \in P(S, S')_{f'}$. The *holonomy group of S over S' based at p* is the image $\mathrm{Hol}_p(S, S') := h_p(L_{f'}(S')) \subset \mathrm{Aut}(S)$.

Recall that the contractible loop group $C_{f'}(S')$ is a normal subgroup of $L_{f'}(S')$ generated by lassos.

Definition 3.2 (Restricted holonomy group)**.** The restricted holonomy group is the image $\mathrm{Hol}_{p0}(S, S') := h_p(L_{f'}(S')) \trianglelefteq \mathrm{Hol}_p(S, S')$.

By Theorem 3.2, the conjugacy classes of both $\mathrm{Hol}_p(S, S')$ and $\mathrm{Hol}_{p0}(S, S')$ are independent of p and of the implicit base face f', so we often omit the initial position p from our notation when discussing general properties of the holonomy group, with the understanding that the discussion is valid for any choice of a base face f' of S' and an initial position $p \in P(S, S')_{f'}$.

Note $\mathrm{Hol}_0(S, S') \trianglelefteq \mathrm{Hol}(S, S')$, since the restricted holonomy group is the image of a normal subgroup under a surjective homomorphism, so we can construct the quotient $Q = \mathrm{Hol}(S, S')/\mathrm{Hol}_0(S, S')$.

Theorem 3.3. *There exists a unique surjective homomorphism $\psi : \pi_1(S') \to Q$ such that the diagram below commutes.*

$$\begin{tikzcd}
0 \arrow[r] & C(S') \arrow[r, hook, "i_1"] \arrow[d, "h|_{C(S')}"] & L(S') \arrow[r, two heads, "q_1"] \arrow[d, "h"] & \pi_1(S') \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
0 \arrow[r] & \mathrm{Hol}_0(S, S') \arrow[r, hook, "i_2"] & \mathrm{Hol}(S, S') \arrow[r, two heads, "q_2"] & Q \arrow[r] & 0
\end{tikzcd}$$

Proof. In the diagram, the top and bottom rows are both short exact sequences: i_1 and i_2 are the inclusion homomorphisms, and q_1 and q_2 are the quotient homomorphisms.

Note that $\mathrm{Ker}(q_1) \subset \mathrm{Ker}(q_2 \circ h)$, since the kernel of q_1 is $C(S')$ and h maps contractible loops into the subgroup $\mathrm{Hol}_0(S, S') \trianglelefteq \mathrm{Hol}(S, S')$, which is precisely the kernel of q_2. Thus, by the universal property of the quotient, there exists a unique homomorphism $\psi : \pi_1(S') \to Q$ such that $q_2 \circ h = \psi \circ q_1$. Furthermore, ψ is surjective because $q_2 \circ h$ is surjective. □

The structure described by the diagram allows us to build up $\mathrm{Hol}(S, S')$ from $\mathrm{Hol}_0(S, S')$ and $\pi_1(S')$. Specifically, having computed $\mathrm{Hol}_0(S, S')$, we can find $\mathrm{Hol}(S, S')$ by picking a representative loop γ from each homotopy class $[\gamma] \in \pi_1(S')$ and computing the coset $h(\gamma)\mathrm{Hol}_0(S, S')$. In other words, we can break down the holonomy group into the restricted holonomy group and the additional cosets of the restricted holonomy group created by nontrivial homotopy classes from the fundamental group. This lets us study holonomy groups by examining restricted holonomy and "fundmanetal group holonomy," which makes computing holonomy groups slightly easier. Fortunately, it's not necessary to check every homotopy class in most cases since we can also use information about the subgroup structure of $\mathrm{Aut}(S)$ to determine $\mathrm{Hol}(S, S')$ by process of elimination.

In fact, we frequently can limit our search not only to subgroups of $\mathrm{Aut}(S)$, but rather the group of orientation-preserving automorphisms of S. We say an automorphism α of a maximally symmetric orientable surface S (without boundary) is orientation-preserving if, given a fixed orientation of S, for each face $\{v_1, v_2, v_3\}$ of S with orientation $[v_1, v_2, v_3]$, the face $\{\alpha(v_1), \alpha(v_2), \alpha(v_3)\}$ has orientation $[\alpha(v_1), \alpha(v_2), \alpha(v_3)]$.

Theorem 3.4. *Let S be a connected, maximally symmetric surface without boundary, and let S' be a connected triangulated surface. If both S and S' are orientable, then* $\mathrm{Hol}(S, S')$ *contains only orientation-preserving automorphisms of S.*

Proof. Pick a position

$$p = \{(v_1, v_1'), (v_2, v_2'), (v_3, v_3')\} \in P(S, S')_{f'}$$

where $f = \{v_1, v_2, v_3\}$ is a face of S and $f' = \{v_1', v_2', v_3'\}$ is a face of S'. Since we assume both S and S' are orientable, without loss of generality, pick orientations of each such that f has orientation $[v_1, v_2, v_3]$ and f' has orientation $[v_1', v_2', v_3']$.

Let $\Phi : \phi(p) \to \phi'(p)$ be the bijection mapping each vertex of f to the vertex of f' with which it is aligned by p. For example, using the position p defined above, $\Phi(v_1) = v_1'$. Given fixed orientations of S and S', we say a position p is orientation-aligning when f has orientation $[v_1, v_2, v_3]$ if and only if f' has orientation $[\Phi(v_1), \Phi(v_2), \Phi(v_3)]$. Given an orientation-aligning position p, the position $\alpha(p)$ is orientation-aligning if and only if $\alpha \in \mathrm{Aut}(S)$ is orientation-preserving.

Since each automorphism in the holonomy group is induced by rolling along a loop (i.e. a sequence of single rolls) from an initial position, it suffices to show that applying a single roll to an orientation-aligning position yields another orientation-aligning position. So, without loss of generality, let $\tilde{f} = \{v_1, v_2, v_4\}$ be the face adjacent to f over edge $e = \{v_1, v_2\}$ on S and likewise let $\tilde{f}' = \{v_1', v_2', v_4'\}$ be a face adjacent to f' over edge $e' = \{v_1', v_2'\}$ on S'. Now let $\rho_{e'} : P(S, S')_{f'} \to P(S, S')_{\tilde{f}'}$ be the rolling map over e', so

$$\rho_{e'}(p) = \{(v_1, v_1'), (v_2, v_2'), (v_4, v_4')\}.$$

By the definition of orientability, the face $\tilde{f}$ of S has orientation $[v_2, v_1, v_4]$ and the face $\tilde{f}'$ of S' has orientation $[v_2', v_1', v_4']$, so $\rho_{e'}(p)$ is orientation-aligning. $\square$

When both S and S' are orientable, this theorem allows us to limit our search for the holonomy group $\mathrm{Hol}(S, S')$ to subgroups of the group of orientation-preserving automorphisms of S, rather than the full automorphism group. This helps significantly, since for maximally symmetric surfaces S, the group of orientation-preserving automorphisms has order $|\mathrm{Aut}(S)|/2 = 3|F|$.

We now restrict our attention to computing $\mathrm{Hol}_0(S, S')$, the restricted holonomy group. Our work here depends on the fact stated earlier that $C(S')$ is generated by lassos. Intuitively, it should be clear that the holonomy induced by rolling along a lasso (up to conjugation) depends only on the loop portion of the lasso, not the path taken from the base face to that loop and back. Formally, let $l = p^{-1} \star l^* \star p$ be a lasso around a vertex v' of S', where p is a path from a face f' to f'^* and l^* is the loop portion of the lasso, i.e. l^* is a loop based at f'^* such that every face in the loop contains v', and such that every face containing v' in S' is included in l^* exactly once (with exception of the f'^*, which must be included twice in order for l^* to be a loop).

Theorem 3.5. *If $l = p^{-1} \star l^* \star p$ is a lasso as defined above, then the automorphism of S induced by l has the same order as the automorphism of S induced by l^*.*

Proof. Using the reasoning of Corollary 3.1, $h(l)$ and $h(l^*)$ are conjugate elements in $\mathrm{Aut}(S)$, and hence have the same order. $\square$

This allows us to state a useful fact that greatly speeds up the process of computing the restricted holonomy group.

Corollary 3.2. *Because we assume S to be maximally symmetric, every vertex of S has the same order, so suppose every vertex of S has degree n. Let l be a lasso on S' around a vertex v'. Then $|h(l)| = n/\gcd(n, \delta(v'))$.*

Proof. By Theorem 3.5, the automorphism $h(l)$ induced by l is conjugate to the automorphism $h(l^*)$ induced by a loop l^* directly around v' (as defined above). In particular, $|h(l)| = |h(l^*)|$.

Rolling S along l^* fixes some vertex v of S at the vertex v' of S'. Moreover, l^* contains exactly $\delta(v')$ rolls, all of which are in the same direction around v'. Thus, rolling S along l^* once will induce the automorphism of S that rotates S around v by $\delta(v')$ steps. Clearly, this automorphism will be the identity if and only if $n|\delta(v')$. In general, the order of $h(l^*)$ will be the minimum natural number k such that $n|k\delta(v')$, that is, $|h(l^*)| = n/\gcd(n, \delta(v'))$. $\square$

In the particular case where S is the tetrahedron, it follows that if l is a lasso on a triangulated surface S' without boundary, then $h(l)$ has order 1 or 3, i.e., $hl(l)$ is either trivial or a rotation about a fixed vertex of the tetrahedron. This is because every vertex of the tetrahedron has degree 3, so clearly if $3|\delta(v')$, then $h(l^*)$ will be trivial. On the other hand, if $3 \nmid \delta(v')$, then $\gcd(3, \delta(v')) = 1$, so $h(l^*)$ will have order 3.

As an illustration of the results given so far, we compute the holonomy group of the tetrahedron S over the small hexagonal torus S' (see the figure).

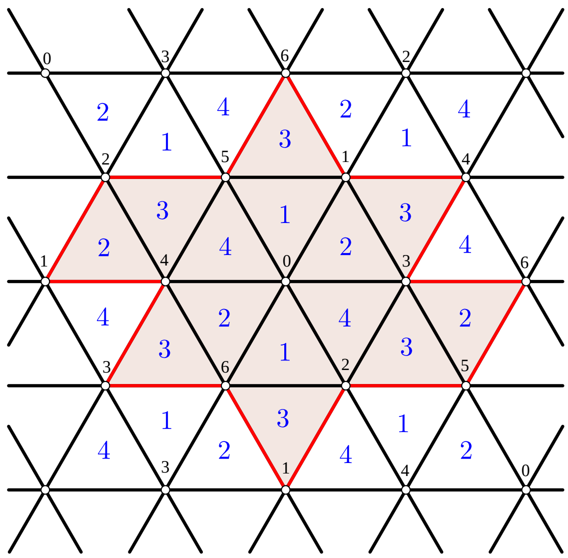


Figure 1: The small hexagonal torus, an orientable triangulated surface without boundary with 7 vertices (numbered here 0-6) and 14 faces.

Every vertex of S' has degree 6, so by Corollary 3.2, $\mathrm{Hol}_0(S, S')$ is trivial, and any non-trivial holonomy must come from the fundamental group. Since $\mathrm{Hol}_0(S, S')$ is trivial, $Q = \mathrm{Hol}(S, S')/\mathrm{Hol}_0(S, S') \cong \mathrm{Hol}(S, S')$, so by Theorem 3.3, ψ maps every homotopy class in the fundamental group to a single element of the holonomy group. Furthermore, the image of ψ (i.e. $\mathrm{Hol}(S, S')$) is determined by where it sends the generators of $\pi_1(S')$. In this case the fundamental group is isomorphic to $\mathbb{Z}^2$, and a quick manual computation using arbitrarily chosen representatives of the two homotopy classes that generate $\pi_1(S')$ shows that ψ maps these generators to distinct elements of order 2 in $\mathrm{Hol}(S, S')$. Since these elements generate the holonomy group, it follows that $\mathrm{Hol}(S, S') \cong V_4$, the Klein four-group.

The methods discussed in this section focus on the relationships between the topological properties of surfaces and their holonomy groups. However, the geometry of the surface (i.e. the particular way in which the surface is triangulated) also affects the holonomy: for instance, the holonomy group of the tetrahedron over a triangulation of the torus with twice as many faces as the triangulation shown in the figure above could be the trivial group, even though both tori are topologically indistinguishable. In the following section, we give some examples of the various holonomy groups that are possible for the tetrahedron over tori and Klein bottles. We then proceed to develop tools to leverage the geometry of a surface to constrain its possible holonomy groups.

4 Determining and Comparing Holonomy Patterns by Adjusting Variables

4.1 Determining Holonomies of Tori with vertices order 6

In this document a torus (plural tori), refers to a triangulated surface without boundary that results from (S_1), the circle, multiplied by itself n times for $n = order of dimension$. In this document all tori are S^3 or S^4.

When laid out on a 2D plane, the flat edges of a hexgrid, comprising of the base of the triangles, are considered the top and bottom. To create the various tori in question, the top and bottom edges are equated and the sides are equated to create a donut-like shape (Figure 2).

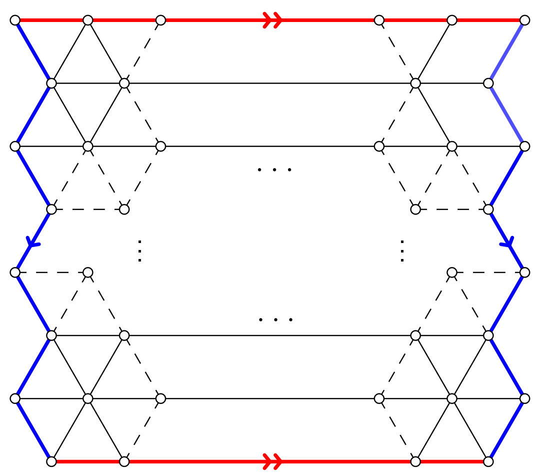


Figure 2: Hexgrid plane with demarcated alignments of sides, which, when aligned, produce a regular Torus

This leads us to the four different ways to create a torus: adjusting the number of rows, adjusting the turning number of either side, and adjusting the length of each row.

Definition 4.1 (Adjusting the number of rows)**.** For a true torus, the number of horizontal rows must be at least 3. If this was not the case, multiple triangles would be defined by the same 3 vertices, because the construction would merely be either two rows back to back or just one row by itself with boundaries. Hence they would disobey the definition of a surface, wherein unique triangles must be defined by three unique vertices (Definition 2.1).

Any torus with $4N$ rows will have trivial holonomy when rolled vertically through the centre of the object.

Hence, when creating a torus, the number of rows can influence the holonomy due to a four row pattern related to the trivial holonomy.

An issue that had to be accounted for when making the torus, was the alignment of the triangles when the top and bottom edges were connected. When the surface had $2M$ rows the edges could easily be connected (Figure 3).

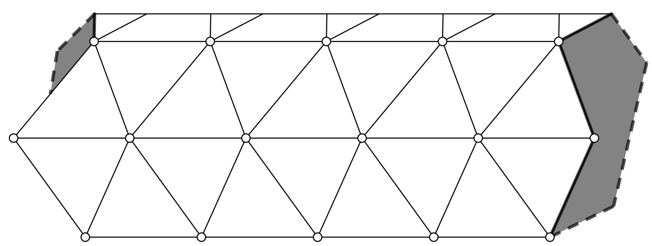


Figure 3: Hexgrid plane wrapped into aligned cylinder

However, if the Torus has $2M + 1$ rows the edges will not align (Figure 4).

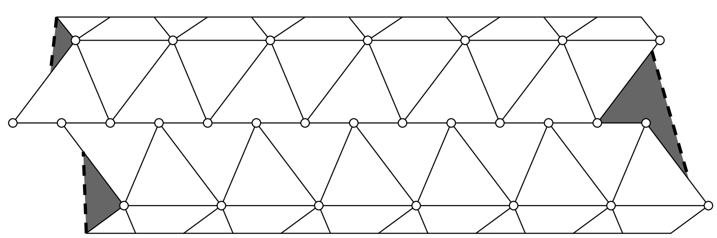


Figure 4: Hexgrid plane with odd number of rows wrapped into cylinder. The triangles do not align.

Definition 4.2 (Adjusting the turning number)**.** If one side is twisted 180 degrees and an experimenter rolls an object along a horizontal path, the rolled object will follow the path it would have on a mobius strip. However, one side does not need to be twisted 180 degrees for a change in holonomy: for a continuous surface one side can be twisted any angle between 0 and ∞. For a combinatorial surface it can be twisted any multiple of 360/(number of rows).

To get unique holonomy, because the number of rows is set in a pattern of length four, the Torus can have no twist, denoted 0; it can have one side twisted so that its second row matches the first row of the other side; have a side twisted so the third row is matched by the first; and finally be twisted so the fourth row is matched by the first. When the rows are twisted so the fifth row matches the first, the four row holonomy pattern repeats and hence is equivalent to no twist, denoted 0. Since the edges of the plane undulate, mismatching by $2K+1$ (any odd numbered rotation) causes ridges to align with ridges and hence cannot be done without increasing or decreasing the order of the faces on the connecting vertices.

Definition 4.3. Adjusting the row length: The length of each row changes the holonomy, because when a tetrahedron is rolled along a horizontal path, it loops through each of the four faces being down. Therefore, if the band length is not 4X, the holonomy when it rolls off the edge of the plane, and hence over onto the other edge, will be adjusted by one face of the tetrahedron.

Similar to an odd numbered rotation, an odd number of triangles in a row will make ridges become troughs and troughs ridges and hence will cause an increase or decrease of the connecting vertex order. An odd numbered rotation combined with an odd numbered row length matches the troughs and ridges and hence results in all vertices being order 6.

Each of the variables have four positions which produce different holonomies. If you go to the fifth position, it is equivalent to the first. Therefore the positions of each variable are listed in modulo 4, also denoted $\mathbb{Z}/4\mathbb{Z}$. In order to classify the changes each $\mathbb{Z}/4\mathbb{Z}$ position all three variables have on the holonomy of a torus, each setting is adjusted whilst the others are kept constant. Because this initial analysis is focused of a regular vertex order of six, odd rotations with odd band lengths are considered one of the basic settings and are included in this list. The different setups are described by "(number of rows, turning number, band length)" where "$(0,0,0)$" is the holonomy of a regular torus with trivial holonomy.

The "Left" and "Right" mean the top side of the net of the hexgrid, combinatorial layout is pulled left or right respectively in order to line up with the bottom side of the net (the issue described in the "number of rows" section previously). Table **??** depicts the various holonomies produced by the different basic setups of a surface.

With these basis positions established we can start to combine them to see the regular changes in holonomy (Table 3).

A clear pattern from the setups is that they are all $\mathbb{Z}/2\mathbb{Z}$ except for the regular, trivial torus. This means that on any chosen face there can be the possibility of two different orientations of the faces of the tetrahedron. When the basic setups are combined they also usually create $\mathbb{Z}/2\mathbb{Z}$ orientations. When two of the same orientations of the basic setups are combined they generally do

Setup (R,T,L)	Holonomy	Permutations
$(0,0,0)$ Left	$\mathbb{Z}/2\mathbb{Z}$	(2,3)
$(0,0,0)$ Right	$\mathbb{Z}/2\mathbb{Z}$	(1,4)
$(1,0,0)$	$\mathbb{Z}/4\mathbb{Z}$	(1,3,4,2)
$(3,0,0)$ Right	$\mathbb{Z}/4\mathbb{Z}$	(1,2,4,3)
$(0,0,2)$	$\mathbb{Z}/4\mathbb{Z}$	(2,3)
(0,1,3)	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{V}$
$(2,3,3)$	$\mathbb{D}_4$	$\mathbb{D}_4$

Table 1: Holonomies of Basic Setups of Klein Bottle

Setup (R,T,L)	Holonomy	Combinations	Permutations
$(1,1,1)$ Left	$\mathbb{Z}/2\mathbb{Z}$	$((1,3)(2,4))^2$	(1,3)(2,4)
$(1,1,3)$ Right	$\mathbb{Z}/2\mathbb{Z}$	$((1,2)(3,4))^2$	(1,2)(3,4)
$(0,2,2)$	Trivial	$((1,4)(2,3))^2$	-
$(2,2,0)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3)^2$	(1,4)(2,3)
$(2,0,2)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3))^2$	(1,4)(2,3)
$(2,3,3)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)(1,3)(2,4)	$\mathbb{V}$

Table 2: Holonomies of Combination Setups of Torus

not change the holonomy, because the same faces are in the same positions no matter which way they are produced.

The two significant arrangements are $(0,2,2)$ and $(2,3,3)$. $(0,2,2)$ has a combination of two 1up, 4right arrangements, but because the surface can be decomposed into a generator vector diagram (Figure 5), wherein the patterns of vertices on the contractible plane repeat at the end of each generator. The turn by two equates the origin with the generator a unit vector directly to the left.

The removal of two triangles creates the same vector. When you add these two vectors they equate to one of the trivial generators pictured, and hence produce trivial holonomy.

The holonomy of $(2,3,3)$ is also interesting because it is $\mathbb{Z}/4\mathbb{Z}$. This results because each of the original setups produce different faces down when the resultant orientations of the $(2,0,0)$ setup is rolled through the $(0,3,3)$ path it produces the 2left orientation. The same happens when the $(0,3,3)$ setup is rolled along the $(2,0,0)$ path.

4.2 Determining Holonomy Patterns of Klein Bottles with vertices order 6

Continuing from Section 4.1, Klein Bottles also produce interesting holonomies with surfaces comprised only of vertices with order 6. A Klein Bottle is a non-orientable surface without boundary. It can be visualised in 3D in Figure 6, but in reality the bottle does not intersect itself.

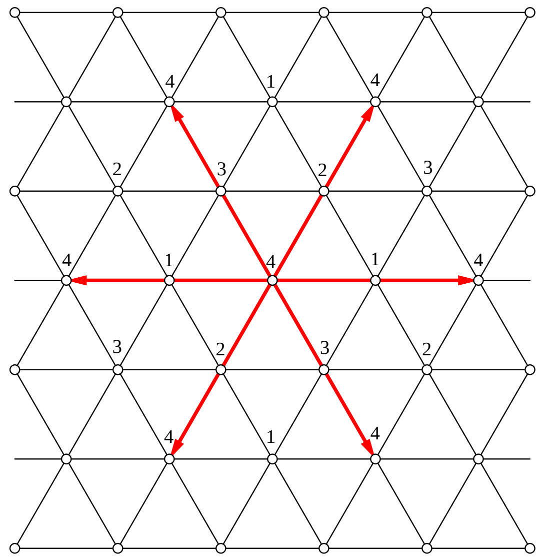


Figure 5: Generators of regular, trivial torus on hexgrid.

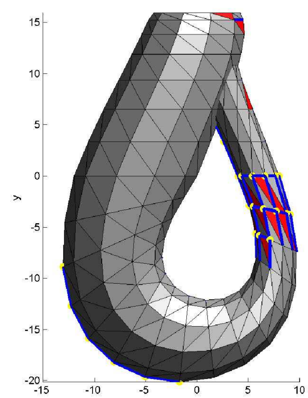


Figure 6: A triangulated 3D visualisation of a Klein Bottle

Similar to a torus, to create the various Klein Bottles in question, the top and bottom are connected to each other and the sides are connected to each other to create a donut-like shape. Unlike a torus, when the sides are connected one side is rotated 180 degrees (Figure 7).

When adjusting the Klein Bottle there are the same three variables as in Section 4.1. In this section however, another variable has been added, which corresponds to the turning number of the flat top and bottom edges.

Definition 4.4 (Adjusting the flat edge turning number)**.** Similarly to the

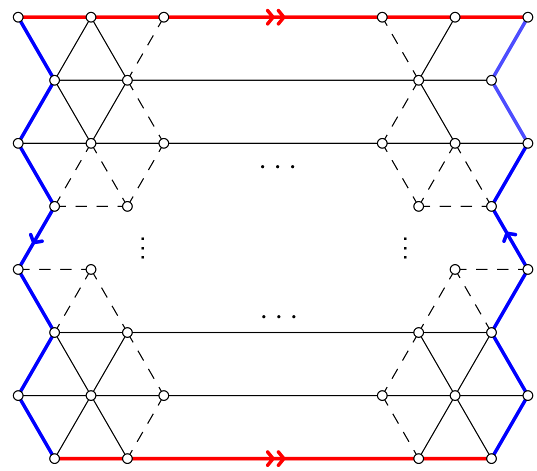


Figure 7: Hexgrid plane with demarcated alignments of sides, which, when aligned, produce a regular Klein Bottle

triangulated edge turning number, the flat edge turning number adjusts the holonomy by aligning the first triangle on the bottom edge with the last triangle on the top edge (the identity case); by aligning the first triangle on the bottom edge with the second to last triangle on the top edge; or any other setup of twists up to ∞. Unlike all other settings we have discusses this setup is in modulo 2, because the pattern of trivial holonomy repeats every turns. This is because each twist actually skips two triangles in each row, since the up triangles only have a single vertex on the top edge, not a triangle edge.

Because diagonally opposite corners are made equivalent in a Klein Bottle, the additional turn caused by different distribution of triangles in alternating rows (Figure 4) is now applied to the even numbered number of rows. When creating a torus, a plane with row length and turn number associated with trivial holonomy of both a torus and a Klein Bottle would associate the edges of the triangle's 0 in each row, which are in the same vertical orientation to each other in an even numbered row (0 and 2). Alternatively, each even numbered row in the plane to create a Klein Bottle would have an overhanging first triangle, and an indented final triangle. When a Klein bottle is created these two will be associated, but the overhang and the indent will not align. In odd numbered rows the first triangle of the first row will be overhanging and the last triangle of the last row will be overhanging, which result in aligned triangles.

The setups of the Klein Bottle planes are denoted '(row number, triangulated edge turning number, row length, flat edge turning number)'.

1,3 are opposite of above

With these basis positions established we can start to combine them to see the regular changes in holonomy (Table **??**).

Setup (R,T,L)	Holonomy	Combinations	Permutations
$(1,1,1)$ Left	$\mathbb{Z}/2\mathbb{Z}$	$((1,3)(2,4))^2$	(1,3)(2,4)
$(1,1,3)$ Right	$\mathbb{Z}/2\mathbb{Z}$	$((1,2)(3,4))^2$	(1,2)(3,4)
$(0,2,2)$	Trivial	$((1,4)(2,3))^2$	-
$(2,2,0)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3)^2$	(1,4)(2,3)
$(2,0,2)$	$\mathbb{Z}/2\mathbb{Z}$	$((1,4)(2,3))^2$	(1,4)(2,3)
$(2,3,3)$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	(1,4)(2,3)(1,3)(2,4)	$\mathbb{V}$

Table 3: Holonomies of Combination Setups of Torus

Setup (R,T,L)	Holonomy	Permutations
$(0,0,0)$ Left	$\mathbb{Z}/2\mathbb{Z}$	(2,3)
$(0,0,0)$ Right	$\mathbb{Z}/2\mathbb{Z}$	(1,4)
$(1,0,0)$	$\mathbb{Z}/4\mathbb{Z}$	(1,3,4,2)
$(3,0,0)$ Right	$\mathbb{Z}/4\mathbb{Z}$	(1,2,4,3)
$(0,0,2)$	$\mathbb{Z}/4\mathbb{Z}$	(2,3)
(0,1,3)	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{V}$
$(2,3,3)$	$\mathbb{D}_4$	$\mathbb{D}_4$

Table 4: Holonomies of Basic Setups of Klein Bottle

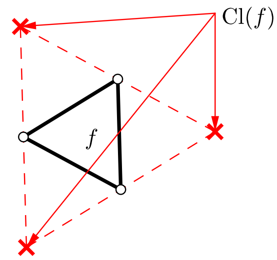
5 Method for Restricting Holonomy

In general, it is fairly easy to find surfaces that have either trivial or full tetrahedral holonomy. It is more difficult to find surfaces that have small but nontrivial holonomy. Above, we gave an example of a surface with tetrahedral holonomy V_4. Now, we turn our attention to finding a method to construct surfaces with tetrahedral holonomy $\mathbb{Z}/3\mathbb{Z}$.

5.1 Restricting holonomy of rolling tetrahedron over orientable surfaces to subgroup of $\mathbb{Z}/3\mathbb{Z}$

Let's consider the holonomy of rolling a tetrahedron S' over a closed, path-connected orientable surface S. We are interested in finding the surfaces with specific holonomy groups such as $\mathbb{Z}/3\mathbb{Z}$ or the trivial holonomy.

Definition 5.1 (Neighborhood of a face)**.** For a face $f \in S$, define the **neighborhood** of f, $\mathrm{Cl}(f)$, to be the set of all vertices of S that do not lie in f, but lie in some faces adjacent to f.



We can define an equivalence relation on the set of vertices of S, such that two vertices v, w are equivalent if and only if there exists a finite rolling that brought a vertex x of the tetrahedron S' which is "standing" on v to the position w (it means that we change from the some position of the form $\{(x, v), \cdots\}$ to $\{(x, w), \cdots\}$). Due to the property of the tetrahedron, if $v \in \mathrm{Cl}(f)$ and $x \in S'$ is standing on v, then by rolling this tetrahedron in simple paths, we can let x reach any vertices in $\mathrm{Cl}(f)$. So essentially, all the possible vertices that x can reach can be identified by expanding continually from v to all the $\mathrm{Cl}(f)$ that contains v, and to all the neighborhood of the faces that contain those vertices we just identified, and so on. Formally, we can define this equivalence relation as followed:

Definition 5.2 (positional equivalence)**.** We say that two vertices v, w are *positional equivalent* if there exists a finite sequence of faces $f_1, f_2, \cdots, f_k$ such that $v \in \mathrm{Cl}(f_1)$, $\mathrm{Cl}(f_i) \cap \mathrm{Cl}(f_{i+1}) \neq \varnothing$ for all $1 \leq i \leq n-1$, and $w \in \mathrm{Cl}(f_k)$. We denote it as $v \simeq w$ and say that v is *positional connected to w by* $(f_1, \cdots, f_k)$.

It is simple to check that this is indeed an equivalence relation. It is obvious that $v \simeq v$ for any $v \in \mathrm{Vert}(S)$. It is also *reflexive*, since if $v \simeq w$, and $(f_1, f_2, \cdots, f_k)$ is a finite sequence of faces that connects v to w, then the inverse sequence $(f_k, f_{k-1}, \cdots, f_1)$ will connect w to v, so $w \simeq v$ as well. Finally, if v is connected to w by $(f_1, \cdots, f_k)$, and w is connected to z by $(f_{k+1}, \cdots, f_n)$, then v is connected to z by $(f_1, \cdots, f_n)$. Hence, $v \simeq w$ and $w \simeq z$ imply that $v \simeq z$, or it is transitive.

Property 5.1. *Let $v, w \in \mathrm{Vert}(S)$, and assume that the initial orientation is $\{(x, v), \cdots\}$. Then there exists a finite rolling that turns $\{(x, v), \cdots\}$ to $\{(x, w), \cdots\}$ if and only if $v \simeq w$.*

Proof. If $v \simeq w$, assume that v is positional connected to w by $(f_1, f_2, \cdots, f_n)$. Let $v_i = \mathrm{Cl}(f_i) \cap \mathrm{Cl}(f_{i+1})$ for $1 \leq i \leq n-1$. We proceed by rolling S' in a path through f_1 moving x from v to v_1, and inductively proceed from v_i to v_{i+1}, until we let x touches w. So we will turn $\{(x, v), \cdots\}$ to $\{(x, w), \cdots\}$ in a sufficient way of rolling.

Assume that there exists a finite rolling that turns $\{(x, v), \cdots\}$ to $\{(x, w), \cdots\}$. Let that roll be $(f_1, f_2, \cdots, f_n)$. □

By defining this equivalence relation, we can classify all the vertices into equivalence classes, each class contains all vertices that are reachable by x if x is standing on one of its vertices. Define an undirected graph $G = (V, E)$ where V is the set of all vertices of S. Two vertices $v \neq w$ are connected by an edge if there exists a face f such that $v, w \in \mathrm{Cl}(f)$. We can see that the equivalence relation defined as above can be interpreted as connectedness in G: $v \sim w$ if and only if v and w are connected by a path in G. The equivalence classes are then the connected components of G.

Lemma 5.1. *Let $v \in S$ be a vertex and assume that S' is standing on a face f of S that contains v. Assume that v is not equivalent to all of its neighbors (which are the vertices that lie on the edges containing v), then the holonomy group is either $\mathbb{Z}/3\mathbb{Z}$ or trivial.*

Proof. Assume that the vertex of S' touching v at the moment is x. We claim that x is the only possible vertex that can reach v after any rolling. Assume the contradiction that any some rolls, a vertex $x' \neq x$ will stand at v. Starting from the initial position, we roll the tetrahedron to a position that x' touches S, and x is standing at v (it's always possible to do this). This means that x' will be standing at a neighbor of v. By our assumption, x' can never reach v since v is not equivalent to any of its neighbors, a contradiction. Hence, after any rolls back to the initial face, we guarantee that the vertex standing at v must be x, and thus the holonomy group can only be either $\mathbb{Z}/3\mathbb{Z}$ or trivial. □

For convenience, we call a vertex v of S *isolated* if it is not equivalent to all of its neighbors. Due to theorem 3.**??**, the holonomy group $H(S, S')$ is independent of the base face, up to an isomorphism. Hence, if there exists an isolated vertex somewhere, then it is sufficient to conclude that the holonomy is a subgroup of $\mathbb{Z}/3\mathbb{Z}$. This gives a helpful tool to restrict the holonomy into only two possible choices by looking locally at a specific vertex or face.

The following claim gives a "sufficient" condition for a surface to have trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy. Assume that we are touching the face $f = \{v_1, v_2, v_3\}$ of S, and assume that the vertices of the tetrahedron touching S' is v_1', v_2', v_3', in particular v_i' touches v_i.

Lemma 5.2. *If the holonomy group is trivial or $\mathbb{Z}/3\mathbb{Z}$, then either one of v_i is isolated or all the vertices in $\mathrm{Cl}(f)$ are isolated.*

Proof. Since the holonomy group is either trivial or $\mathbb{Z}/3\mathbb{Z}$, then there must either exist vertex v_i of S that fixed the vertex v_i' of S' (which means after any rolling in a loop, it can only be v_i' that touches v_i), or the vertex not touching S is fixed in that position. Assume the first case, we will show that v_i is isolated. Assume the contrary that v_i is not isolated, then there is w adjacent to v_i such that $v_i \sim w$. We roll the tetrahedron to the face containing both w, v_i. Then, since $w \sim v_i$, there exists a loop that rolling the tetrahedron through that loop will change v_i' from standing at v_i to standing at w. At this moment, a vertex $v \neq v_i'$ of the tetrahedron S' will stand at v_i, and then by rolling S' to the original face, we get a position that v_i' does not stand at v_i, which is contrary to our assumption. Hence, v_i is isolated.

Assume the second case, let v be the vertex not touching the surface S. Since when we roll back to f in any paths, we must bring v to the untouching position, and thus this is only possible if and only if v is the only vertex that can touch the vertices in $\mathrm{Cl}(f)$, and thus all of them must be isolated. □

Theorem 5.3. *The holonomy group is a subgroup of $\mathbb{Z}/3\mathbb{Z}$ if and only if the face it's standing on has at least one isolated vertex or the neighborhood of it contains an isolated vertex.*

Proof. Assume that we are standing on f of S. If f has an isolated vertex, then it follows by Lemma 5.1 that the holonomy is a subgroup of $\mathbb{Z}/3\mathbb{Z}$. If $\mathrm{Cl}(f)$ has an isolated vertex, then by Theorem 3.**??**, the holonomy group is isomorphic to the holonomy when we starts at the face contains that isolated vertex, which is $\mathbb{Z}/3\mathbb{Z}$ due to Lemma 5.1.

Assume now that the holonomy group is $\mathbb{Z}/3\mathbb{Z}$, by Lemma 5.2, either f has an isolated vertex, or all the vertices of $\mathrm{Cl}(f)$ are isolated, which concludes the reverse direction. □

Corollary 5.1. *The holonomy group is a subgroup of $\mathbb{Z}/3\mathbb{Z}$ if and only if S has an isolated vertex.*

The theorem and collorary are useful because they helps us restrict the holonomy group to two only possible choices. Also, to generalize these claims to non-orientable surfaces, instead of saying that the holonomy is a subgroup of $\mathbb{Z}/3\mathbb{Z}$, we use the fact that the existence of an isolated vertex is equivalent to the holonomy group being generated by rotations around the fixed vertex that touches the isolated vertex in the beginning. By using this, the holonomy group is either trivial, $\mathbb{Z}/3\mathbb{Z}$, or S_3, and conversely if the holonomy is one of those, then the initial position has an isolated vertex.

Back to orientable surfaces, this condition alone can not indicate whether the holonomy group is $\mathbb{Z}/3\mathbb{Z}$ or trivial. However, the above two claims give us an idea to strengthen the condition of the vertexes that gives us a trivial holonomy group.

Theorem 5.4. *The holonomy group is trivial if and only if the three vertices of the face it's standing on are all isolated.*

Proof. The reverse direction is clear. Now, assume that the holonomy group is trivial, and assume the contradiction that one of v_i is not isolated, WLOG let it be v_1. Let $w \neq v_1$ be a neighbor of v_1 such that $w \sim v_i$. By the similar argument in Claim 2, there exists a roll in a loop that moves a vertex $v \neq v_i' \in S'$ to touch v_1, and thus gives us a non-trivial holonomy, a contradiction. □

Corollary 5.2. *S has trivial holonomy if and only if all vertices of S are isolated*

Now we consider which possible surfaces give us the full rotation symmetry group or the Klein four-group V_4. By the claims, this can't happen if we are standing on a face that has an isolated vertex. But even if we assume that the surface S has no isolated vertices, or even that every vertex of S lies in the same equivalence class, we can not conclude that the holonomy is A_4. Indeed, one counterexample of this is the hexagonal torus. Since the holonomy group is V_4, we can check that any vertex of the tetrahedron S' can reach any vertices of S, and we have only one equivalence class. So looking at these equivalence classes of vertices alone will not tell us anything about whether the holonomy group is V_4 or A_4. However, it gives a quite sufficient indication of the trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy group.

5.2 Subdivision

In this following section, we present an application of the above results to indicate the holonomy of a specific kind of surfaces - subdivided surfaces.

Definition 5.3 (Subdivision)**.** Let S be a triangulated surface. An n-subdivision of S is a surface S_n obtained by doing the following steps:

1. For each edge $\{a, b\}$ of S, we add $n-1$ "boundary" vertices

$$a_1b_{n-1}, a_2b_{n-2}, \cdots, a_{n-1}b_1.$$

   If we are considering the edge $\{a, b\}$ in the context of the face $\{a, b, c\}$, for convenience we can also notate $a_ib_{n-i} \equiv a_ib_{n-i}c_n$, and $a = a_0b_nc_n$.

2. On each face $\{a, b, c\}$ of S, we add "internal" vertices $a_ib_jc_k$ for $2 \leq i, j, k \leq n-1$ such that $i + j + k = 2n$.

3. Replace each face $\{a, b, c\}$ of S with n^2 faces determined by one of these two forms:

$$\{a_ib_jc_k, a_{i+1}b_{j-1}c_k, a_{i+1}b_jc_{k-1}\}$$

   for $1 \leq j, k \leq n, 0 \leq i \leq n-1$ and $i + j + k = 2n$, or

$$\{a_ib_jc_k, a_ib_{j+1}c_{k-1}, a_{i+1}b_jc_{k-1}\}$$

   for $0 \leq i, j \leq n-1, 1 \leq k \leq n$ and $i + j + k = 2n$. (where $a_0b_nc_n = a, a_nb_0c_n = b, a_nb_nc_0 = c$)

To make sense of this formal definition, imagine physically that for each edge of the triangular face $f = \{a, b, c\}$, for example $\{b, c\}$, we draw $n + 1$ lines $a_0, a_1, a_2, \cdots, a_n$ parallel to that edge where $a_n = \{b, c\}$, a_0 is the line passing through a, and $a_1, a_2, \cdots$ are the lines moving far from a in this order. We choose them in a way that they divide the edges $\{a, b\}, \{a, c\}$ into equal segments, where the intersections on $\{a, b\}$ are denoted to be $a_1b_{n-1}, a_2b_{n-2}, \cdots, a_{n-1}b_1$. Define b_i, c_i for $0 \leq i \leq n$ and a_ic_{n-i}, b_ic_{n-i} for $1 \leq i \leq n-1$ similarly. We can see that three mutually nonparallel lines a_i, b_j, c_k are then concur if and only if

$i+j+k=2n$, and we define the concurrence point as $a_ib_jc_k$. If one of i,j,k is n, then this point will lie on the boundary edges of f. If there are two of i,j,k equal to n, then it is one of the original vertices a,b,c. Otherwise, we define it to be internal. Then, the identification of the new triangulation using these points are specified as above, where the "upward pointing" triangle has the form $\{a_ib_jc_k, a_{i+1}b_{j-1}c_k, a_{i+1}b_jc_{k-1}\}$, and "downward pointing" triangle has the form $\{a_ib_jc_k, a_ib_{j+1}c_{k-1}, a_{i+1}b_jc_{k-1}\}$.

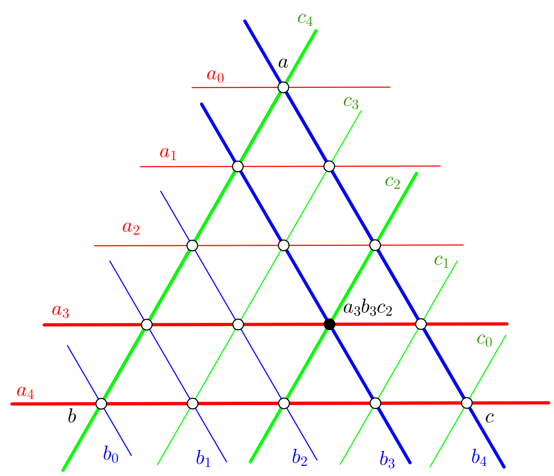


Figure 8: An example of 4−subdivided triangulated face

An $n-$subdivision of a surface is obtained by subdividing every face in this way. It can be easily showed that a subdivided surface is closed, and furthermore, all boundary and internal vertices have degrees 6. For convenience, we denote $\mathrm{Vert}(S)$ as the set of all $S-$vertices, and $F(S)$ the set of all $S-$faces.

Property 5.2. $\mathrm{Vert}(S_n) = \mathrm{Vert}(S) \cup A$ *where* $A \cap \mathrm{Vert}(S) = \varnothing$ *and* A *is a set of vertices, all have degrees* 6*. Furthermore, the degrees of all vertices in* $\mathrm{Vert}(S)$ *are preserved through subdivision.*

Lemma 5.5. *For any integers* $m,n>1$*, we have* $S_{mn} = (S_m)_n = (S_n)_m$*.*

Proof. It suffices to show that $S_{mn} = (S_m)_n$. Consider the $mn-$subdivision S_{mn} defined as above. Consider the surface S' with vertices $V \subset \mathrm{Vert}(S_{mn})$, where

$$V = \{a_{ni}b_{nj}c_{nk} \mid \{a,b,c\} \in F(S), 0 \le i,j,k \le m, i+j+k=2m\}$$

and the set of faces F of S is defined by two forms:

1. $\{a_{ni}b_{nj}c_{nk}, a_{n(i+1)}b_{n(j-1)}c_{nk}, a_{n(i+1)}b_{nj}c_{n(k-1)}\}$ for $1 \le j,k \le m, 0 \le i \le m-1$ and $i+j+k=2m$

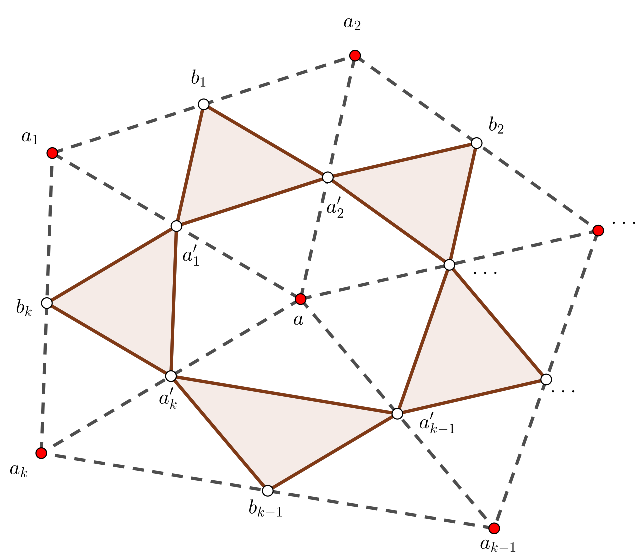
2. $\{a_{ni}b_{nj}c_{nk}, a_{ni}b_{n(j+1)}c_{n(k-1)}, a_{n(i+1)}b_{nj}c_{n(k-1)}\}$ for $0 \le i, j \le m-1, 1 \le k \le m$ and $i+j+k = 2m$.

It can be easily checked by the definition that $S' = S_m$. Now, we can also check that S_{mn} is the $n-$subdivision of S_m by checking that every face of one of the above form can also be $n-$subdivided. Hence, $S_{mn} = (S_m)_n$. $\square$

Define a graph $G(V, E)$ where $V = \mathrm{Vert}(S_n)$, and we connect two vertices of V if and only if they belong to the same **neighborhood** $\mathrm{Cl}(f)$ of some face f. For this section, we assume that S is an orientable, path-connected surface. It can be shown by checking the definition that S_2 is also orientable.

Theorem 5.6. *S_2 has either a trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy.*

Proof. We will show that in the graph G, no vertices of $\mathrm{Vert}(S)$ are connected to any vertices of A. To show this, we will simply show that for an arbitrary $a \in \mathrm{Vert}(S)$, all the neighbors of a in G are also in $\mathrm{Vert}(S)$. Let $a_1, a_2, \cdots, a_k$ be the adjacent vertices of a in the surface S (i.e. they are the vertices in the linkage of a with respect to the surface S), and let $a'_1, a'_2, \cdots, a'_k$ be the adjacent vertices of a in the subdivided surface S_n, such that for each $1 \le i \le k$, a'_i is adjacent to a_i and a. For each $1 \le i \le k$, let b_i be the unique vertex such that b_i is adjacent to both a_i, a_{i+1} (where $a_{k+1} \equiv a_1$).



Since there are exactly k faces of S_n that contains a, a lies in exactly k neighborhoods of k different faces, in particular, $\{a'_1, a'_2, b_1\}, \cdots, \{a'_{k-1}, a'_k, b_{k-1}\}$, $\{a'_k, a'_1, b_k\}$. For each $1 \le i \le k$, we have $\mathrm{Cl}(\{a'_i, a'_{i+1}, b_i\}) = \{a, a_i, a_{i+1}\}$ (we can check this easily by listing the faces adjacent to $\{a'_i, a'_{i+1}, b_i\}$, using the formal definition of subdivision as above, but it might be too detailed and distracted so I use a visualization to make it clear). We can see that all the vertices that lie in the same neighborhoods of some faces with a can only be $a_1, a_2, \cdots, a_k$, which are vertices of $\mathrm{Vert}(S)$.

So by showing this, we can conclude that every vertex in $\mathrm{Vert}(S)$ is isolated, because it is only equivalent to other vertices in $\mathrm{Vert}(S)$, but their neighbors in the subdivided S_n are all vertices in $\mathrm{Vert}(S_n) \setminus \mathrm{Vert}(S)$. Hence, by Collorary 5.1, the holonomy is a subgroup of $\mathbb{Z}/3\mathbb{Z}$. $\square$

Corollary 5.3. *If n is even, then S_n has either a trivial or $\mathbb{Z}/3\mathbb{Z}$ holonomy.*

Proof. Let $n = 2k$. Since $S_n = (S_k)_2$, and S_k is closed, then S_n is a $2-$subdivision of a closed surface S_k. By Theorem 5.6, the holonomy of S_n is a subgroup of $\mathbb{Z}/3\mathbb{Z}$. $\square$

Generally, for any surface S (orientable or not), subdividing it by an even n will turn all vertices of S into isolated vertices. From this, we can conclude the holonomy group, depends on whether it is orientable or not.

Turning back to orientable S, we can actually distinguish more whether S_2 has the holonomy $\mathbb{Z}/3\mathbb{Z}$ or trivial with this following claim.

Lemma 5.7. *If S is a finite, connected surface having a vertex with a degree not divisible by 3, then all the vertices in $\mathrm{Vert}(S_2) \setminus \mathrm{Vert}(S)$ are equivalent.*

Proof. Let $a \in \mathrm{Vert}(S)$ be a vertex with a degree not divisible by 3. We will reuse the notations from Theorem 5.6. We have for any i, $a'_i, a'_{i+3} \in \mathrm{Cl}(\{a, a'_{i+1}, a'_{i+2}\})$, since $\{a, a'_i, a'_{i+1}\}, \{a, a'_{i+2}, a'_{i+3}\}$ are adjacent to $\{a, a'_{i+1}, a'_{i+2}\}$. Thus, $a'_i \simeq a'_{i+3}$. Since $3 \nmid \delta(a)$, for any i, there exists k such that $k\delta(a) + i \equiv 1 \pmod 3$. So we will have $a'_1 \simeq a'_{3h+1} \simeq a'_{k\delta(a)+i} \equiv a'_i$, so all a'_i are equivalent.

Also, note that since $\{b_i, a'_i, a'_{i+1}\}, \{a, a'_{i+1}, a'_{i+2}\}$ are adjacent to $\{a, a'_i, a'_{i+1}\}$, we have $b, a'_{i+2} \in \mathrm{Cl}(\{a, a'_i, a'_{i+1}\})$, so $b_i \simeq a'_{i+2} \simeq a'_1$ for any i, and all b_i are equivalent to a'_i as well.

From this, we can see that each a_i has three equivalent neighbors, namely b_i, b_{i+1}, a'_i. By the same reasoning as above, each neighbor of a_i must be equivalent to one of these consecutive vertices, and since they are all equivalent, all neighbors of a_i are also equivalent. Then, we can repeat the same argument for a above to all a_i, and then to all the neighbors of a_i in S, and so on, until it spreads to all the vertices of S, which is possible since S is finite and connected. From this, we can conclude that all the vertices of $\mathrm{Vert}(S_2) \setminus \mathrm{Vert}(S)$ are equivalent. $\square$

Theorem 5.8. *If S has a vertex with degree not divisible by 3, then S_2 has holonomy $\mathbb{Z}/3\mathbb{Z}$.*

Proof. Assume that it has trivial holonomy, then by Corollary 5.2, it has all vertices isolated. However, this contradicts Lemma 5.7. Hence it must be $\mathbb{Z}/3\mathbb{Z}$ due to Theorem 5.6. $\square$

It is also easy to generalize this claim to any even n, noting that a subdivision preserves degrees of the original vertices, and S_{2k} is simply a $2-$subdivision of S_k.

5.3 Restricting holonomy to non-orientable using equivalence classes

Recall from the previous results, we have shown that the holonomy when rolling the tetrahedron over a surface that has an isolated vertex is a subgroup of $\mathbb{Z}_3$. The idea to generalize to any surfaces (orientable or not), is to notice that the existence of an isolated vertex is equivalent to having a holonomy group that fixes a vertex. It means that the holonomy is a subgroup of S_n that contains the permutations that fix a specific vertex x of the tetrahedron.

Theorem 5.9. *S has an isolated vertex if and only if the holonomy group fixes a vertex of the tetrahedron*

Proof. Assume that S has an isolated vertex v and the tetrahedron is in a position $p = \{(x, v), \cdots\}$, and assume for the contradiction that there exists a rolling of the tetrahedron in some loop that we get a position $p' = \{(x', v), \cdots\}$ for $x' \neq x$. Then, either x is touching one of v's neighbors, or x is not touching the surface S at the moment, but after a roll we can bring x to touch any neighbor of v. In either ways, there is a neighbor of v that is equivalent to v, which contradicts the assumption that v is an isolated vertex.

The inverse part is pretty trivial, since if v is not isolated, then there will be a neighbor $v' \simeq v$ of v, and then we roll to a position $\{(x, v'), \cdots\}$, and then we can roll to a position that x is not the vertex that touches v, which gives the contradiction. □

Note that since the tetrahedron is a maximally symmetric surface, we can conclude about the holonomy group based on the holonomy that we derive locally. So we have an important corollary:

Corollary 5.4. *S has an isolated vertex if and only if the holonomy group is a subgroup of S_3.*

Note that if S is orientable, it's impossible to have $\mathbb{Z}_2, S_3$, and inversely, if S is non-orientable, then it's impossible to have trivial or $\mathbb{Z}_3$ holonomies, since $\mathbb{Z}_3$ subgroup of S_3 contain only even permutations, but if S isnon-orientable then it must have an odd permutation in the holonomy. We develop criteria for non-orientable surfaces that have $\mathbb{Z}_2$ and S_3 holonomy. Assume that S has an isolated vertex (note that this assumption is really important).

Criterion 1 ($\mathbb{Z}_2 - (a, b)$ holonomy)**.** S has $\mathbb{Z}_2$ holonomy if and only if it has two adjacent isolated vertices, but not every vertex of S is isolated.

Remark. If S has two adjacent isolated vertices, but not every vertex of S is isolated, then S is non-orientable.

Remark. If S is orientable and it has two isolated adjacent vertices, then all vertices of S are isolated and S has trivial holonomy.

Criterion 2 (S_3 holonomy)**.** S has S_3 holonomy if and only if it has an isolated vertex that has all neighbors not isolated.

So to sum up this part, we have this table to indicate holonomy group for surfaces that have at least one isolated vertex.

Holonomy group	Orientable	Non-orientable
Trivial	all vertices are isolated	don't exist
$\mathbb{Z}/2\mathbb{Z}$	don't exist	there are two adjacent isolated vertices, but not all vertices are isolated.
$\mathbb{Z}/3\mathbb{Z}$	not all vertices are isolated, but there is one.	don't exist
S_3	don't exist	there is an isolated vertex that has all neighbors not isolated.

Table 5: Criteria for identifying holonomy groups